Turk. J. Math. Comput. Sci.
17(2)(2025) 322–337
© MatDer
DOI : 10.47000/tjmcs.1729279

Topological Generalizations of Graphs: Integrating Hypergraph and SuperHyperGraph Perspectives

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Received: 28-06-2025 • Accepted: 18-09-2025

Abstract. A hypergraph is a generalization of a graph in which edges—called hyperedges—can connect any number of vertices, not just two. A SuperHyperGraph further extends this framework by introducing recursive powerset constructions, enabling the representation of hierarchical relationships among hyperedges themselves.
A *topological graph* is a graph embedded in the plane, where each vertex is represented by a distinct point and each edge is drawn as a continuous curve connecting its endpoints. A *topological hypergraph* extends this idea by representing hyperedges as closed curves that enclose sets of vertices on the plane [13].
In this paper, we introduce the concept of a *Topological n-SuperHyperGraph*, which unifies the structural hierarchy of *n*-SuperHyperGraphs with the geometric intuition of topological hypergraphs. We anticipate that this new formulation will contribute to future developments in graph theory, topology, and network theory.

2020 AMS Classification: 05C65

Keywords: SuperHyperGraph, hypergraph, topological hypergraph, topology, topological graph, topological superHyperGraph.

1. Introduction

1.1. **Graph, Hypergraph, and SuperHyperGraph.** Graph theory studies structures made of vertices and edges, modeling relationships, networks, and connections in various scientific fields [19, 35]. Network modeling most commonly starts with simple graphs, in which vertices represent objects and edges denote pairwise connections. To accommodate interactions involving more than two elements, hypergraphs allow each edge to link any nonempty subset of vertices, thereby capturing higher-order relationships [10, 20, 21]. Building on this idea, a *SuperHyperGraph* employs repeated powerset operations to represent nested, hierarchical linkages among vertices [31, 55, 57]. This multi-layered approach has attracted growing interest in recent studies [32, 47].

Table 1 summarizes the principal differences between graphs, hypergraphs, and SuperHyperGraphs. Throughout this work, *n* denotes a nonnegative integer unless otherwise specified.

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Concept	Notation	Edge Type	Extension Mechanism
Graph	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Edges connect exactly two vertices.
Hypergraph	$H = (V, E)$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Hyperedges join arbitrary nonempty subsets of vertices.
SuperHyperGraph	$\mathrm{SuHyG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathcal{P}^n(V_0), E \subseteq \mathcal{P}^n(V)$	Uses an n-fold powerset to encode nested structure.

TABLE 1. Key distinctions among graphs, hypergraphs, and SuperHyperGraphs.

1.2. **Topology and Graph.** A topology on a set is a collection of subsets—called *open sets*—that is closed under arbitrary unions and finite intersections [45]. A *topological graph* is a graph drawn in the plane in which each vertex is a distinct point and each edge is a continuous curve (Jordan arc) connecting its endpoints; edges are allowed to intersect, but only at finitely many points [43]. This model is central to network theory and to many practical systems representable as graphs [18, 39]. In particular, topological graphs capture connectivity up to continuous deformation, supporting deformation-invariant reasoning, robust surface embeddings, homotopy-based arguments, and compatibility with topological data analysis (TDA)—useful for sensor networks, robotics, routing, structural analysis, and noise-resilient modeling. In artificial intelligence, related ideas underlie *topological graph neural networks*, an active research area [38, 44, 66]. A *topological hypergraph* extends this paradigm by realizing hyperedges as simple closed curves (e.g., circles) in the plane; a hyperedge contains precisely the vertices lying in the corresponding closed region [7, 13]. Such realizations geometrically encode higher-arity relations and enable deformation-invariant analysis, intuitive visualization, homotopy-theoretic tools, and surface-aware algorithms for complex multiway interactions.

1.3. **Our Contribution.** As discussed above, research on Graph Theory, Hypergraph Theory, SuperHyperGraph Theory, Topological Graphs, and Topological Hypergraphs is of significant importance. However, research on Topological SuperHyperGraphs has not yet been sufficiently developed. To bridge this gap, we propose the following contributions. In this paper, we introduce and formalize the concept of a Topological SuperHyperGraph, which has received little attention so far. We investigate its mathematical structure and its relationship with existing graph-based frameworks. We hope that our work will contribute to the advancement of Graph Theory, Topological Graph Theory, Network Theory, and their various applications.

Table 2 summarizes the key distinctions among topological graphs, topological hypergraphs, and topological SuperHyperGraphs. The concept of topological SuperHyperGraphs is newly introduced and formally defined in this paper.

Concept	Topological Embedding	Key Feature	References
Topological Graph	Vertices as points in $\mathbb{R}^2$; edges as continuous injective arcs	Edges may cross finitely many times; intersections only at points	[49, 61]
Topological Hypergraph	Vertices as points; hyperedges as simple closed (Jordan) curves	Each curve encloses exactly the vertices in that hyperedge	[7, 13]
Topological SuperHyperGraph	Elements of $\mathcal{P}^n(P)$ realized as nested regions in $\mathbb{R}^2$	Recursive hierarchy: n-supervertices and n-superedges drawn as nested closed sets	This Paper

TABLE 2. Comparison of topological graph, topological hypergraph, and topological SuperHyperGraph.

1.4. **Structure of This Paper.** This section outlines the structure of the paper. Section 2 provides the definitions of key concepts such as Hypergraphs, SuperHyperGraphs, Topological Graphs, and Topological Hypergraphs. Section 3 introduces the notion of the Topological SuperHyperGraph and discusses its properties. Section 4 presents the conclusion of the paper and outlines possible directions for future work.

2. PRELIMINARIES

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. For fundamental operations, concepts, and principles of graphs, refer to [19, 42].

2.1. **Powerset and n-th Powerset.** The powerset of a set is the collection of all possible subsets, including the empty set and the set itself. The nth powerset is created by taking the powerset operation n times in succession, resulting in nested layers of subsets.

Definition 2.1 ([40]). [Set] A *set* is a collection of distinct objects, called elements, which are unambiguously defined. If A is a set and x is an element of A, we write $x \in A$. Sets are usually denoted by enclosing their elements in curly braces.

Definition 2.2 ([40]). [Subset] For any two sets A and B, A is said to be a *subset* of B (written $A \subseteq B$) if every element of A is also an element of B:

$$\forall x \in A, \quad x \in B.$$

If additionally $A \neq B$, then A is called a *proper subset* of B, denoted $A \subset B$.

Definition 2.3 ([40]). [Empty Set] The *empty set*, denoted $\varnothing$, is the unique set that contains no elements:

$$\forall x, \quad x \notin \varnothing.$$

It follows that $\varnothing$ is a subset of every set.

Definition 2.4 ([33]). [Base Set] A *base set* S is the underlying set from which more elaborate structures, such as powersets and hyperstructures, are constructed. It is defined by

$$S = \{x \mid x \text{ belongs to a specified domain}\}.$$

All elements appearing in constructions like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ are drawn from S.

Definition 2.5 ([40, 51]). [Powerset] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including both $\varnothing$ and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.6 (n-th Powerset). (cf. [41,56,58,59]) The n-th powerset of a set H, denoted $P_n(H)$, is defined recursively by:

$$P_1(H) = \mathcal{P}(H), \quad P_{n+1}(H) = \mathcal{P}(P_n(H)) \quad \text{for } n \geq 1.$$

Similarly, the n-th nonempty powerset, denoted $P_n^*(H)$, is given by:

$$P_1^*(H) = \mathcal{P}^*(H), \quad P_{n+1}^*(H) = \mathcal{P}^*(P_n^*(H)),$$

where $\mathcal{P}^*(H)$ denotes the powerset of H with the empty set omitted.

Example 2.7 (Real-World Illustration of the n-th Powerset). Let $D = \{\text{Soup, Salad, Steak}\}$ be the set of dishes available for a single meal. Then:

$$P_1(D) = \mathcal{P}(D) = \{\varnothing, \{\text{Soup}\}, \{\text{Salad}\}, \{\text{Steak}\}, \{\text{Soup,Salad}\}, \{\text{Soup,Steak}\}, \{\text{Salad,Steak}\}, \{\text{Soup,Salad,Steak}\}\}.$$

Each element of $P_1(D)$ represents a possible meal combination.

To plan an entire week, we select one menu for each of the seven days. These "weekly plans" form the second powerset:

$$P_2(D) = \mathcal{P}(P_1(D)),$$

where each element of $P_2(D)$ is a set of seven (not necessarily distinct) daily menus. For example,

$$\{\{\text{Salad}\}, \{\text{Soup,Steak}\}, \ldots\} \in P_2(D)$$

might encode "Salad on Monday, Soup + Steak on Tuesday," and so on.

More generally, $P_n(D)$ models an n-level planning process (e.g. daily, weekly, monthly, etc.), where each level chooses subsets from the previous one.

2.2. **Graphs and Hypergraphs.** In classical graph theory, a *hypergraph* generalizes the concept of a standard graph by allowing an edge—called a *hyperedge*—to connect more than two vertices [19]. This expanded structure makes hypergraphs particularly suitable for modeling complex relationships among elements, thereby increasing their applicability in various fields such as computer science, biology, and network theory [8, 9, 21, 34]. In the following, we provide precise definitions of hypergraphs. Throughout this paper, we restrict our attention to finite, undirected, and simple graphs.

Definition 2.8 (Hypergraph)**.** [10, 12] A *hypergraph* $H = (V(H), E(H))$ consists of:

- A nonempty set $V(H)$ of vertices.
- A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Example 2.9 (Hypergraph)**.** Let the vertex set be

$$V = \{v_1, v_2, v_3, v_4\},$$

and define the hyperedge set by

$$E = \Big\{\{v_1, v_2, v_3\},\ \{v_2, v_4\}\Big\}.$$

Then the hypergraph

$$H = (V, E)$$

is a example in which one hyperedge connects three vertices while the other connects two.

Example 2.10 (Market Basket as a Hypergraph)**.** Let the set of products be

$$V(H) = \{\text{Milk}, \text{Bread}, \text{Eggs}, \text{Butter}, \text{Cereal}\}.$$

Each customer transaction is modeled as a hyperedge (a nonempty subset of products purchased together). For instance,

$$E(H) = \{\{\text{Milk}, \text{Bread}\}, \{\text{Milk}, \text{Eggs}, \text{Butter}\}, \{\text{Bread}, \text{Cereal}\}, \{\text{Milk}, \text{Bread}, \text{Eggs}\}\}.$$

Then, $H = (V(H), E(H))$ is a finite hypergraph in which each hyperedge records a multi-item co-purchase. Unlike standard graphs (which only capture pairwise co-occurrences), this hypergraph directly represents higher-order purchase relationships among several products in a single basket.

2.3. **SuperHyperGraph.** A *SuperHyperGraph* is a higher–order generalization of the classical hypergraph obtained by allowing vertices themselves to live on iterated powerset levels. This framework has been developed and studied in recent works [15, 22, 31, 32, 57]. In recent years, SuperHyperGraphs have attracted growing interest for potential applications in decision–making [36, 37], graph neural networks (GNNs) [26, 30, 33], medicine [29], chemistry [24, 28], quantum theory [27], energy systems [23], urban elderly care [14], media convergence [69], business [11, 63], engineering [25], and others [16, 50, 64]. The definition and concrete examples of an n-SuperHyperGraph are given below.

Definition 2.11 ([54, 55])**.** [n-SuperHyperGraph] Let V_0 be a finite nonempty base set (of *atomic vertices*). Define the iterated powerset by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}(\mathcal{P}^k(V_0)) \quad (k \geq 0),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset and $\mathcal{P}^*(\cdot)$ the nonempty powerset. An *n–SuperHyperGraph* is a pair

$$\text{SuHyG}^{(n)} \ = \ (V, E)$$

such that

$$V \ \subseteq \ \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \ \subseteq \ \mathcal{P}^*(V).$$

Elements of V are called *n-supervertices* (each is an element on level n over V_0), and elements of E are *n-superedges* (each is a nonempty subset of V). For $n = 0$ this recovers the usual hypergraph ($V \subseteq V_0,\ E \subseteq \mathcal{P}^*(V)$).

Example 2.12 (Organization Structure as a 2–SuperHyperGraph)**.** Let

$$V_0 = \{\text{Ayuka}, \text{Masahiro}, \text{Carol}, \text{Dave}\}.$$

Teams are elements of the first powerset $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$. Choose three teams

$$T_1 = \{\text{Ayuka}, \text{Masahiro}\}, \quad T_2 = \{\text{Carol}, \text{Dave}\}, \quad T_3 = \{\text{Ayuka}, \text{Carol}\}.$$

Form 2 – level supervertices as singleton *team–clusters*:

$$V = \{\{T_1\}, \{T_2\}, \{T_3\}\} \subseteq \mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0)).$$

A division that groups all three clusters is represented by the 2 – superedge

$$e = \{\{T_1\}, \{T_2\}, \{T_3\}\} \in \mathcal{P}^*(V), \qquad E = \{e\}.$$

Then, $\text{SuHyG}^{(2)} = (V, E)$ is a 2 – SuperHyperGraph in which superedges are subsets of the set V of 2 – supervertices, ensuring incidence is well–typed at the same level.

Example 2.13 (3–SuperHyperGraph for an IT Product–Line Hierarchy)**.** Let the base modules be

$$V_0 = \{\text{Authentication}, \text{Payment}, \text{Analytics}\}.$$

Level 1 (projects): $M_1 = \{\text{Authentication}, \text{Payment}\}$, $M_2 = \{\text{Payment}, \text{Analytics}\}$, $M_3 = \{\text{Authentication}, \text{Analytics}\}$ in $\mathcal{P}^1(V_0)$. Level 2 (releases): $P_A = \{M_1, M_2\}$, $P_B = \{M_2, M_3\}$ in $\mathcal{P}^2(V_0)$. Level 3 (portfolios): $L_1 = \{P_A, P_B\}$, $L_2 = \{P_B\}$ in $\mathcal{P}^3(V_0)$.

Take the 3 – supervertex set and a single 3 – superedge by

$$V = \{L_1, L_2\} \subseteq \mathcal{P}^3(V_0), \qquad E = \{\{L_1, L_2\}\} \subseteq \mathcal{P}^*(V).$$

Thus, $\text{SuHyG}^{(3)} = (V, E)$ encodes the hierarchy

$$\text{modules} \longrightarrow \text{projects} \longrightarrow \text{releases} \longrightarrow \text{portfolios},$$

with superedges connecting 3-level supervertices (portfolios).

Example 2.14 (3–SuperHyperGraph: Multi-Hospital Clinical Initiatives)**.** Let the base set of clinicians be

$$V_0 = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}, \text{Erin}\}.$$

Level 1 (clinical teams) consists of subsets of V_0:

$$T_1 = \{\text{Alice}, \text{Bob}\}, \quad T_2 = \{\text{Carol}, \text{Dave}\}, \quad T_3 = \{\text{Bob}, \text{Erin}\} \in \mathcal{P}^1(V_0).$$

Level 2 (care programs) groups teams:

$$P_A = \{T_1, T_2\}, \qquad P_B = \{T_2, T_3\} \in \mathcal{P}^2(V_0).$$

Level 3 (clinical initiatives) groups programs:

$$I_1 = \{P_A\}, \qquad I_2 = \{P_A, P_B\} \in \mathcal{P}^3(V_0).$$

Define the 3–supervertex set and a single 3–superedge by

$$V = \{I_1, I_2\} \subseteq \mathcal{P}^3(V_0), \qquad E = \{\{I_1, I_2\}\} \subseteq \mathcal{P}^*(V).$$

Then, $\text{SuHyG}^{(3)} = (V, E)$ models a hospital network in which

$$\text{clinicians} \rightarrow \text{teams} \rightarrow \text{programs} \rightarrow \text{initiatives}.$$

Example 2.15 (4–SuperHyperGraph: Smart-Campus IoT Deployment)**.** Let the base set of IoT devices be

$$V_0 = \{\text{SensorA}, \text{SensorB}, \text{SensorC}, \text{SensorD}\}.$$

Level 1 (rooms: device groups):

$$R_1 = \{\text{SensorA}, \text{SensorB}\}, \quad R_2 = \{\text{SensorB}, \text{SensorC}\}, \quad R_3 = \{\text{SensorC}, \text{SensorD}\} \in \mathcal{P}^1(V_0).$$

Level 2 (buildings: room groups):

$$B_1 = \{R_1, R_2\}, \qquad B_2 = \{R_2, R_3\} \in \mathcal{P}^2(V_0).$$

Level 3 (campuses: building groups):

$$C_1 = \{B_1\}, \qquad C_2 = \{B_1, B_2\} \in \mathcal{P}^3(V_0).$$

Level 4 (regions: campus groups):

$$\mathsf{Reg}_{\mathrm{North}} = \{C_1\}, \qquad \mathsf{Reg}_{\mathrm{Central}} = \{C_1, C_2\} \in \mathcal{P}^4(V_0).$$

Set

$$V = \{\mathsf{Reg}_{\mathrm{North}}, \mathsf{Reg}_{\mathrm{Central}}\} \subseteq \mathcal{P}^4(V_0), \qquad E = \{\{\mathsf{Reg}_{\mathrm{North}}, \mathsf{Reg}_{\mathrm{Central}}\}\} \subseteq \mathcal{P}^*(V).$$

Thus, $\mathrm{SuHyG}^{(4)} = (V, E)$ captures the hierarchy

$$\text{devices} \rightarrow \text{rooms} \rightarrow \text{buildings} \rightarrow \text{campuses} \rightarrow \text{regions},$$

with 4–supervertices as regions and superedges connecting them.

2.4. **Topological Graphs and Topological HyperGraph.** We describe Topological Graphs and Topological Hypergraphs using the following definitions. A topological graph is a planar depiction of a graph where vertices are distinct points and edges are continuous curves [2, 17, 43, 49, 61]. A topological hypergraph extends this idea by using simple closed curves (such as circles) in the plane to define hyperedges. Each hyperedge consists of the vertices enclosed by one of these curves [7, 13].

Definition 2.16 ([1, 4, 48]). [Topological Graph] A *topological graph* is a pair

$$G = (P, E),$$

where:

- $P \subset \mathbb{R}^2$ is a finite set of *vertices* (points in the plane).
- E is a finite set of *edges*. Each edge $e \in E$ is given by a continuous injective mapping

  $$e : [0, 1] \rightarrow \mathbb{R}^2$$

  such that $e(0), e(1) \in P$ (the endpoints) and, except at the endpoints, the image $e((0, 1))$ does not contain any vertex from P.

We allow edges to cross but require that if two edges do not share a vertex then they intersect in at most finitely many points (typically in proper crossings).

Example 2.17 (Urban Road Network as a Topological Graph). Consider a simple urban road network with four intersections:

$$P = \{A(0, 0), B(2, 0), C(0, 2), D(2, 2)\} \subset \mathbb{R}^2.$$

We model the roads by the following continuous, injective mappings:

$$\begin{aligned}
&e_{AB} : [0, 1] \rightarrow \mathbb{R}^2, \quad e_{AB}(t) = (2t, 0), && \text{road from } A \text{ to } B,\\
&e_{AC} : [0, 1] \rightarrow \mathbb{R}^2, \quad e_{AC}(t) = (0, 2t), && \text{road from } A \text{ to } C,\\
&e_{AD} : [0, 1] \rightarrow \mathbb{R}^2, \quad e_{AD}(t) = (2t, 2t), && \text{diagonal street from } A \text{ to } D,\\
&e_{BC} : [0, 1] \rightarrow \mathbb{R}^2, \quad e_{BC}(t) = (2 - 2t, 2t), && \text{diagonal street from } B \text{ to } C.
\end{aligned}$$

Here, $e_{AB}(0) = e_{AC}(0) = e_{AD}(0) = A$ and similarly for other endpoints; all curves avoid other vertices except at their endpoints. The curves e_{AD} and e_{BC} cross at $(1, 1)$ without sharing a vertex, illustrating a proper crossing.

Thus

$$G = (P, E), \quad E = \{e_{AB}, e_{AC}, e_{AD}, e_{BC}\},$$

is a topological graph representing this road network.

Definition 2.18 ([7, 13]). [Topological Hypergraph] A *topological hypergraph* is a pair

$$H = (P, C),$$

where:

- $P \subset \mathbb{R}^2$ is a finite set of *vertices*.

- C is a finite collection of simple closed curves (i.e. Jordan curves) in $\mathbb{R}^2$ that are disjoint from P. For each $C \in C$, denote by
$$P_C = \{x \in P \mid x \text{ lies in the bounded region } \operatorname{disc}(C)\},$$
where $\operatorname{disc}(C)$ is the bounded region determined by C (by the Jordan Curve Theorem). We then view P_C as a *hyperedge*.

Thus, a topological hypergraph can be alternatively described as
$$H' = \Big(P, \{P_C : C \in C\}\Big).$$

Example 2.19 (A Topological Graph as a Topological Hypergraph)**.** Let
$$P = \{(0,0), (1,0), (0,1)\} \subset \mathbb{R}^2,$$
and define the topological graph $G = (P, E)$ where the edges are given by Jordan arcs connecting the points. For instance, let
$$e_1 : [0,1] \to \mathbb{R}^2, \quad e_1(t) = (t, 0)$$
connect $(0,0)$ and $(1,0)$. For e_1, choose an auxiliary arc a_1 that connects $(1,0)$ to $(0,0)$ along a path outside the set P (say, an arc lying in $\{(x,y) : y < -0.1\}$). Then the union $C_{e_1} = e_1([0,1]) \cup a_1([0,1])$ is a Jordan curve and
$$P_{C_{e_1}} = \{(0,0), (1,0)\}.$$
Thus, G is embedded into the topological hypergraph $H = (P, \{C_{e_1}, \ldots\})$.

Example 2.20 (A Topological Hypergraph with a Hyperedge of Cardinality 3)**.** Let
$$P = \{(0,0), (1,0), (0,1), (1,1)\} \subset \mathbb{R}^2,$$
and consider a Jordan curve C which is, for instance, a circle that surrounds exactly the three points $(0,0)$, $(1,0)$, and $(0,1)$ but not $(1,1)$. Then the hyperedge corresponding to C is
$$P_C = \{(0,0), (1,0), (0,1)\}.$$
Thus, $H = (P, \{C\})$ is a topological hypergraph where one hyperedge connects three vertices.

Proposition 2.21 (Topological Hypergraphs Generalize Topological Graphs)**.** *Let $G = (P, E)$ be a topological graph. Then there exists a topological hypergraph*
$$H = \Big(P, \{C_e : e \in E\}\Big)$$
such that for each edge $e \in E$ there is a corresponding simple closed curve C_e satisfying
$$P_{C_e} = \{e(0), e(1)\}.$$
In other words, every topological graph may be regarded as a topological hypergraph in which every hyperedge has cardinality 2.

Proof. For each edge $e \in E$, let $e : [0,1] \to \mathbb{R}^2$ be the continuous injection with endpoints $e(0)$ and $e(1)$ in P. Since $e([0,1])$ is an arc connecting $e(0)$ and $e(1)$, one can choose an auxiliary Jordan arc $a_e : [0,1] \to \mathbb{R}^2$ such that:

- $a_e(0) = e(1)$ and $a_e(1) = e(0)$,
- a_e does not pass through any vertex in P except its endpoints,
- the concatenation
$$C_e = e([0,1]) \cup a_e([0,1])$$
is a simple closed curve.

By the Jordan Curve Theorem, C_e divides $\mathbb{R}^2$ into two regions; denote by $\operatorname{disc}(C_e)$ the bounded one. By choosing a_e appropriately (for instance, so that the only points of P in $\operatorname{disc}(C_e)$ are $e(0)$ and $e(1)$), we have
$$P_{C_e} = \{x \in P : x \text{ is in } \operatorname{disc}(C_e)\} = \{e(0), e(1)\}.$$
Thus, by setting $C = \{C_e : e \in E\}$, we obtain the topological hypergraph $H = (P, C)$ which generalizes G. □

3. Main Results of This Paper

As the main contribution of this paper, we define the notion of *Topological n-SuperHyperGraphs* and investigate their structural properties.

3.1. **Topological n-SuperHyperGraphs.** In this subsection, we examine the fundamental characteristics and mathematical features of Topological n-SuperHyperGraphs. A topological n-SuperHyperGraph integrates iterated powerset vertices and edges with topological realizations, modeling hierarchical geometric relationships.

Definition 3.1 (Topological n-SuperHyperGraph)**.** Let $P \subset \mathbb{R}^2$ be a finite set (called the *base set*). A *topological n-SuperHyperGraph* is a pair

$$\mathrm{SuHyG}^{(n)} = (V, E),$$

where

- $V, E \subseteq \mathcal{P}^n(P)$. In other words, both the vertices and the hyperedges are selected from the nth iterated powerset of P.
- There exists a topological realization of the elements of V and E. For example, one may associate to each $v \in V$ a subset of $\mathbb{R}^2$ (or a collection of points) and to each $e \in E$ a corresponding topological object (such as a closed region) such that the *incidence relation* is defined by the natural inclusion in the hierarchy of the powerset.

In particular, when $n = 1$ we recover that $V, E \subseteq \mathcal{P}(P)$ so that every hyperedge is simply a subset of P; hence, a topological 1-SuperHyperGraph is exactly a topological hypergraph.

A real-world example of a Topological n-SuperHyperGraph is provided below.

Example 3.2 (Urban Planning as a Topological 2-SuperHyperGraph)**.** Let

$$P = \{M, A, P\} = \{\text{Museum}(0,0),\ \text{ArtGallery}(2,0),\ \text{Park}(1,2)\} \subset \mathbb{R}^2$$

be three city landmarks. Define two neighborhood boundaries by simple closed curves:

$$C_1\text{: encloses } \{M, A\}, \quad C_2\text{: encloses } \{A, P\}.$$

By the Jordan Curve Theorem, these curves determine hyperedges in the topological hypergraph on P:

$$P_{C_1} = \{M, A\}, \quad P_{C_2} = \{A, P\}.$$

These form two 1–hyperedges in $\mathcal{P}(P)$.

To build a 2–SuperHyperGraph, consider

$$P_2(P) = \mathcal{P}(\mathcal{P}(P)).$$

Choose the 2–supervertices

$$V = \{\{P_{C_1}\}, \{P_{C_2}\}\} \subseteq P_2(P),$$

each representing one neighborhood cluster, and define a city zone boundary C_Z that encloses both neighborhoods. This curve yields the 2–superedge

$$E = \{\{P_{C_1}, P_{C_2}\}\} \subseteq P_2(P).$$

By assigning to each $\{P_{C_i}\}$ the region inside C_i and to $\{P_{C_1}, P_{C_2}\}$ the region inside C_Z, we obtain a valid topological realization. Hence,

$$\mathrm{SuHyG}^{(2)} = (V, E)$$

is a Topological 2–SuperHyperGraph modeling the nested structure of neighborhoods within a city zone.

Example 3.3 (LAN/WAN Topology as a Topological 2–SuperHyperGraph)**.** Let

$$P = \{\text{LA1}(0,0),\ \text{LA2}(2,0),\ \text{LB1}(0,2),\ \text{LB2}(2,2)\} \subset \mathbb{R}^2$$

represent four routers in two Local Area Networks (LANs). We define two Jordan curves in the plane:

$$C_{\mathrm{LAN_A}} \text{ encloses } \{\text{LA1}, \text{LA2}\}, \quad C_{\mathrm{LAN_B}} \text{ encloses } \{\text{LB1}, \text{LB2}\}.$$

By the Jordan Curve Theorem, each curve yields a topological hyperedge:

$$P_{C_{\mathrm{LAN_A}}} = \{\text{LA1}, \text{LA2}\}, \quad P_{C_{\mathrm{LAN_B}}} = \{\text{LB1}, \text{LB2}\}.$$

These lie in $\mathcal{P}(P)$. To model the Wide Area Network (WAN) bridging both LANs, let C_{WAN} be a simple closed curve that encloses the regions bounded by $C_{\mathrm{LAN_A}}$ and $C_{\mathrm{LAN_B}}$. Then,

$$P_{C_{\mathrm{WAN}}} = \{\, P_{C_{\mathrm{LAN_A}}},\ P_{C_{\mathrm{LAN_B}}} \} \ \in\ \mathcal{P}(\mathcal{P}(P)).$$

We now set

$$V = \{\{P_{C_{\mathrm{LAN_A}}}\},\ \{P_{C_{\mathrm{LAN_B}}}\}\} \subseteq \mathcal{P}^2(P), \quad E = \{\{P_{C_{\mathrm{LAN_A}}}, P_{C_{\mathrm{LAN_B}}}\}\} \subseteq \mathcal{P}^2(P).$$

Thus,

$$\mathrm{SuHyG}^{(2)} = (V, E)$$

is a Topological 2–SuperHyperGraph capturing the nested structure of two LAN segments within a WAN.

Example 3.4 (Cellular Network as a Topological 2–SuperHyperGraph)**.** Let

$$P = \{\mathrm{T_A}(0,0),\ \mathrm{T_B}(2,0),\ \mathrm{T_C}(0,2),\ \mathrm{T_D}(2,2)\} \subset \mathbb{R}^2$$

be four cell-tower locations. Define two simple closed curves representing individual "cells":

$$C_{\mathrm{Cell1}} \text{ encloses } \{\mathrm{T_A}, \mathrm{T_B}\}, \quad C_{\mathrm{Cell2}} \text{ encloses } \{\mathrm{T_C}, \mathrm{T_D}\}.$$

By the Jordan Curve Theorem, these determine the hyperedges

$$P_{C_{\mathrm{Cell1}}} = \{\mathrm{T_A}, \mathrm{T_B}\}, \quad P_{C_{\mathrm{Cell2}}} = \{\mathrm{T_C}, \mathrm{T_D}\} \ \in\ \mathcal{P}(P).$$

To model the entire metropolitan network, let C_{Metro} be a simple closed curve that encloses both C_{Cell1} and C_{Cell2}. Then

$$P_{C_{\mathrm{Metro}}} = \{\, P_{C_{\mathrm{Cell1}}},\ P_{C_{\mathrm{Cell2}}}\} \ \in\ \mathcal{P}(\mathcal{P}(P)).$$

We set

$$V = \{\{P_{C_{\mathrm{Cell1}}}\},\ \{P_{C_{\mathrm{Cell2}}}\}\} \subseteq \mathcal{P}^2(P), \quad E = \{\{P_{C_{\mathrm{Cell1}}}, P_{C_{\mathrm{Cell2}}}\}\} \subseteq \mathcal{P}^2(P).$$

Thus,

$$\mathrm{SuHyG}^{(2)} = (V,\ E)$$

is a Topological 2–SuperHyperGraph capturing the nested structure of two cellular coverage areas within a metropolitan network.

Example 3.5 (Urban Neighborhoods and Districts as a Topological 3–SuperHyperGraph)**.** Let

$$P = \{\, H,\ F,\ S,\ L\} = \{\mathrm{Hospital}(0,0),\ \mathrm{FireStation}(2,0),\ \mathrm{School}(0,2),\ \mathrm{Library}(2,2)\} \subset \mathbb{R}^2$$

be four key landmarks in a city.

First level (topological hyperedges): Define three simple closed curves in the plane:

$$C_{HF} \text{ encloses } \{H, F\}, \quad C_{FS} \text{ encloses } \{F, S\,\}, \quad C_{SL} \text{ encloses } \{S, L\}.$$

By the Jordan Curve Theorem, each C_{XY} induces a hyperedge

$$P_{C_{XY}} = \{X, Y\} \ \in\ \mathcal{P}(P).$$

Second level (2–supervertices): Let

$$P_2(P) = \mathcal{P}(\mathcal{P}(P)).$$

Choose two 2–supervertices representing neighborhood clusters:

$$N_A = \{\, P_{C_{HF}},\ P_{C_{FS}}\,\}, \quad N_B = \{\, P_{C_{FS}},\ P_{C_{SL}}\,\}.$$

Each N_* is an element of $P_2(P)$.

Third level (3–supervertices): Form two city districts as subsets of $P_2(P)$:

$$D_1 = \{\, N_A\}, \quad D_2 = \{\, N_B\}.$$

Thus,

$$V = \{\, D_1,\ D_2\} \ \subseteq\ \mathcal{P}^3(P).$$

Third-level hyperedge (topological 3–superedge): Draw a larger simple closed curve C_{City} in $\mathbb{R}^2$ that encloses both neighborhood boundaries C_{N_A} and C_{N_B} (where C_{N_A} and C_{N_B} are the regions determined by the curves around N_A and N_B). This produces the 3–superedge

$$E = \{\ \{\, D_1,\ D_2\}\ \} \ \subseteq\ \mathcal{P}^3(P).$$

Topological realization: Each first-level curve C_{XY} encloses the two landmarks X, Y. Each second-level neighborhood N_A, N_B is realized by the union of the corresponding first-level regions. The third-level city curve C_{City} encloses both neighborhood regions, realizing the single 3–superedge.

Therefore,

$$\mathrm{SuHyG}^{(3)} = (V,\ E)$$

is a valid Topological 3–SuperHyperGraph modeling the hierarchical structure of landmarks → neighborhoods → districts within a city.

Example 3.6 (Topological 3–SuperHyperGraph: Urban Park Zoning and Events)**.** Let the base point set of amenities in a city park be

$$P = \{p_1 = (0,0),\ p_2 = (2,0),\ p_3 = (0,2),\ p_4 = (2,2),\ p_5 = (1,1)\} \subset \mathbb{R}^2.$$

Level 1 (amenity clusters) in $\mathcal{P}^1(P) = \mathcal{P}(P)$:

$$A_1 = \{p_1, p_2\}, \quad A_2 = \{p_3, p_4\}, \quad A_3 = \{p_5\}.$$

Level 2 (zonal groupings) in $\mathcal{P}^2(P) = \mathcal{P}(\mathcal{P}(P))$:

$$Z_1 = \{A_1, A_3\}, \qquad Z_2 = \{A_2\}.$$

Level 3 (event layouts) in $\mathcal{P}^3(P) = \mathcal{P}(\mathcal{P}^2(P))$:

$$L_1 = \{Z_1\}, \qquad L_2 = \{Z_1, Z_2\}.$$

Take the 3–supervertex set and a single 3–superedge by

$$V = \{L_1, L_2\} \subseteq \mathcal{P}^3(P), \qquad E = \{\{L_1, L_2\}\} \subseteq \mathcal{P}(V).$$

Topological realization. Associate to each point $p_i \in P$ the singleton $\{p_i\} \subset \mathbb{R}^2$. For each A_j (level 1), realize it as the finite point set $\bigcup_{p\in A_j}\{p\}$. For each Z_k (level 2), take the closed convex hull of the union of its member points (a closed polygon in $\mathbb{R}^2$). For each L_ℓ (level 3), realize it as a closed region containing the polygons of all $Z \in L_\ell$. Finally, realize the 3–superedge $\{L_1, L_2\}$ as a larger closed region containing the realizations of both L_1 and L_2. Incidence is by inclusion across levels, matching the definition of a topological 3–SuperHyperGraph.

Example 3.7 (Topological 4–SuperHyperGraph: Maritime Monitoring Command Structure)**.** Let coastal sensors be

$$P = \{q_1 = (0,0),\ q_2 = (3,0),\ q_3 = (0,3),\ q_4 = (3,3),\ q_5 = (1.5,1)\} \subset \mathbb{R}^2.$$

Level 1 (patrol routes; sets of sensors) in $\mathcal{P}^1(P)$:

$$R_1 = \{q_1, q_2, q_5\}, \qquad R_2 = \{q_3, q_4\}.$$

Level 2 (sectors; sets of routes) in $\mathcal{P}^2(P)$:

$$S_A = \{R_1\}, \qquad S_B = \{R_1, R_2\}.$$

Level 3 (districts; sets of sectors) in $\mathcal{P}^3(P)$:

$$D_{\mathrm{North}} = \{S_A, S_B\}, \qquad D_{\mathrm{South}} = \{S_B\}.$$

Level 4 (commands; sets of districts) in $\mathcal{P}^4(P)$:

$$C_1 = \{D_{\mathrm{North}}\}, \qquad C_2 = \{D_{\mathrm{North}}, D_{\mathrm{South}}\}.$$

Let

$$V = \{C_1, C_2\} \subseteq \mathcal{P}^4(P), \qquad E = \{\{C_1, C_2\}\} \subseteq \mathcal{P}(V).$$

Topological realization. Realize each sensor q_i as the point $\{q_i\} \subset \mathbb{R}^2$. Each route R_j is realized as its point set. Each sector $S_\bullet$ is realized by the closed convex hull of the union of its routes (a closed polygon). Each district $D_\star$ is realized by a closed region containing the polygons of its sectors. Each command $C_\diamond$ is realized by a larger closed region containing its districts. The 4–superedge $\{C_1, C_2\}$ is realized as a closed region containing both command regions. Thus incidence is given by geometric inclusion, yielding a topological 4–SuperHyperGraph.

Theorem 3.8 (Topological n-SuperHyperGraphs are n-SuperHyperGraphs)**.** *Let $P \subset \mathbb{R}^2$ be a finite base set. Then the structure*

$$\mathrm{SuHyG}^{(n)} = (V, E)$$

with

$$V, E \subseteq \mathcal{P}^n(P)$$

together with the natural incidence (set inclusion) and an appropriate topological realization, satisfies the axioms of an n-SuperHyperGraph.

Proof. By definition, an n-SuperHyperGraph is a pair (V, E) with $V, E \subseteq \mathcal{P}^n(P)$ where the elements (called n-supervertices and n-superedges) are obtained by applying the power set operator n times. In our topological version, we assume that there is a mapping

$$\varphi : \mathcal{P}^n(P) \to \mathcal{T}(\mathbb{R}^2)$$

which assigns to each subset (or "cluster") a topological object in $\mathbb{R}^2$ (e.g., a set of points or a region) and which preserves the inclusion structure.

Since the operations (such as forming the powerset) are defined recursively by

$$\mathcal{P}^0(P) = P, \quad \mathcal{P}^{k+1}(P) = \{A \mid A \subseteq \mathcal{P}^k(P)\},$$

the pair (V, E) naturally inherits the structure of an n-SuperHyperGraph. In particular, the incidence relation is given by the set membership relation in the nth power set. Hence, the axioms are satisfied by construction. □

Theorem 3.9 (Topological n-SuperHyperGraphs Generalize Topological Hypergraphs)**.** *Let $H = (P, C)$ be a topological hypergraph with*

$$C \subset \{C \mid C \text{ is a Jordan curve in } \mathbb{R}^2\},$$

and let P be identified with the base set. Then, for any integer $n \geq 1$, the hypergraph H can be embedded into the topological n-SuperHyperGraph

$$\mathrm{SuHyG}^{(n)} = (V, E)$$

by identifying each hyperedge P_C with an element of $\mathcal{P}(P) \subset \mathcal{P}^n(P)$ (since for $n \geq 1$ one has $\mathcal{P}(P) \subset \mathcal{P}^n(P)$). In particular, topological graphs (the case of hyperedges of size 2) are a special case.

Proof. Starting with $H = (P, C)$, note that each hyperedge is given by

$$P_C = \{x \in P \mid x \text{ is inside } \operatorname{disc}(C)\},$$

so that $P_C \in \mathcal{P}(P)$. Since for any $n \geq 1$ we have the natural inclusion

$$\mathcal{P}(P) \subseteq \mathcal{P}^n(P),$$

we can view the set of hyperedges $\{P_C : C \in C\}$ as a subset of $\mathcal{P}^n(P)$.

Similarly, we identify the vertex set P with a subset of $\mathcal{P}^n(P)$ via the inclusion

$$P \cong \{\{x\} : x \in P\} \subset \mathcal{P}(P) \subset \mathcal{P}^n(P).$$

Thus, the pair

$$(\{\{x\} : x \in P\}, \{P_C : C \in C\})$$

defines a substructure of the n-SuperHyperGraph $\mathrm{SuHyG}^{(n)}$. Hence, every topological hypergraph (and thus every topological graph as a hypergraph with edges of size 2) is naturally a topological n-SuperHyperGraph. □

Theorem 3.10 (Level-Reduction Theorem)**.** *Let $\mathrm{SuHyG}^{(n)} = (V, E)$ be a topological n-SuperHyperGraph on base set $P \subset \mathbb{R}^2$. For any $1 \leq k \leq n$, define the projection*

$$\pi_k : \mathcal{P}^n(P) \longrightarrow \mathcal{P}^k(P)$$

by "forgetting" the last $n - k$ iterations of the powerset. Then

$$V_k = \pi_k(V), \quad E_k = \pi_k(E)$$

produce a topological k-SuperHyperGraph $\mathrm{SuHyG}^{(k)} = (V_k, E_k)$.

Proof. By construction $V_k, E_k \subseteq \mathcal{P}^k(P)$. A topological realization of each $v \in V_k$ and $e \in E_k$ is obtained by inheriting the region or point–collection from the original realization of any preimage under π_k. Incidence is preserved because set-inclusion in $\mathcal{P}^n(P)$ descends to inclusion in $\mathcal{P}^k(P)$. Hence (V_k, E_k) satisfies the definition of a topological k-SuperHyperGraph. □

Theorem 3.11 (Monotonicity of Substructures)**.** *If $\mathrm{SuHyG}^{(n)} = (V, E)$ is a topological n-SuperHyperGraph and $V' \subseteq V$, then (V', E) is also a topological n-SuperHyperGraph on the same base set.*

Proof. Clearly $V' \subseteq \mathcal{P}^n(P)$ and $E \subseteq \mathcal{P}^n(P)$. The original topological realization of each element of V' and E remains valid. Since no new incidence relations are introduced, (V', E) still meets all axioms. □

Theorem 3.12 (Homeomorphism Invariance)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism mapping $P \subset \mathbb{R}^2$ onto another finite set P'. If $\mathrm{SuHyG}^{(n)} = (V, E)$ is realized by regions and point-sets in $\mathbb{R}^2$, then*

$$\mathrm{SuHyG}'^{(n)} = (\ f(V),\ f(E)\ )$$

is a topological n-SuperHyperGraph on P', where

$$f(V) = \{\{\, f(x) : x \in v\} \mid v \in V\}, \quad f(E) = \{\{\, f(x) : x \in e\} \mid e \in E\}.$$

Proof. Since f is bijective and continuous with continuous inverse, it sends each Jordan region or point-set realizing an n-supervertex or n-superedge to another such object. The preserved inclusion relations between these objects guarantee that $(f(V), f(E))$ satisfies the definition of a topological n-SuperHyperGraph on P'. □

Theorem 3.13 (Disjoint Union Theorem)**.** *If $\mathrm{SuHyG}_1^{(n)} = (V_1, E_1)$ on base P_1 and $\mathrm{SuHyG}_2^{(n)} = (V_2, E_2)$ on disjoint base P_2 are topological n-SuperHyperGraphs, then their union*

$$\mathrm{SuHyG}_1^{(n)} \ \sqcup\ \mathrm{SuHyG}_2^{(n)} = (\ V_1 \cup V_2,\ E_1 \cup E_2\ )$$

is a topological n-SuperHyperGraph on $P_1 \cup P_2$.

Proof. Since $P_1 \cap P_2 = \emptyset$, we have $\mathcal{P}^n(P_1) \cap \mathcal{P}^n(P_2) = \emptyset$, so $V_1 \cup V_2$ and $E_1 \cup E_2$ lie in $\mathcal{P}^n(P_1 \cup P_2)$. Their realizations occupy disjoint regions in $\mathbb{R}^2$, and no unwanted incidences appear between elements from different components. Thus the union structure meets all requirements. □

Theorem 3.14 (Embedding of First-Level Skeleton)**.** *Let $\mathrm{SuHyG}^{(n)} = (V, E)$ be a topological n-SuperHyperGraph on P. Define the 1-skeleton*

$$V^{(1)} = \{\{x\} : x \in P\}, \quad E^{(1)} = \{\, \{u, v\} : \{u, v\} \subseteq e \text{ for some } e \in E\}.$$

Then $(V^{(1)}, E^{(1)})$ is a topological graph on P and naturally embeds into $\mathrm{SuHyG}^{(n)}$.

Proof. By definition $V^{(1)}, E^{(1)} \subseteq \mathcal{P}(P)$, and each edge $\{u, v\}$ arises from some topological region realizing a superedge e. Tracing the boundary of that region between u and v yields a Jordan arc, so we obtain a continuous embedding of the graph $(V^{(1)}, E^{(1)})$ in the plane. Inclusion of $\mathcal{P}(P) \subseteq \mathcal{P}^n(P)$ shows it sits inside the full SuperHyperGraph. □

Theorem 3.15 (Level-Extension Theorem)**.** *Let $\mathrm{SuHyG}^{(k)} = (V_k, E_k)$ be a topological k-SuperHyperGraph on base set $P \subset \mathbb{R}^2$, and let $n \geq k$. Then*

$$V_n = V_k, \quad E_n = E_k$$

viewed as subsets of $\mathcal{P}^n(P) \supseteq \mathcal{P}^k(P)$, define a topological n-SuperHyperGraph $\mathrm{SuHyG}^{(n)} = (V_n, E_n)$.

Proof. Since $\mathcal{P}^k(P) \subseteq \mathcal{P}^n(P)$, we have $V_n, E_n \subseteq \mathcal{P}^n(P)$. The original Jordan regions or point-collections realizing each element of V_k and E_k still serve as a valid topological realization in $\mathbb{R}^2$. Incidence (set inclusion) and continuity requirements are unchanged by this inclusion, so (V_n, E_n) meets the definition of a topological n-SuperHyperGraph. □

Theorem 3.16 (Intersection Theorem)**.** *Let $\mathrm{SuHyG}_1^{(n)} = (V_1, E_1)$ and $\mathrm{SuHyG}_2^{(n)} = (V_2, E_2)$ be two topological n-SuperHyperGraphs on the same base P. Then*

$$V = V_1 \cap V_2, \quad E = E_1 \cap E_2$$

define a topological n-SuperHyperGraph $\mathrm{SuHyG}^{(n)} = (V, E)$.

Proof. Since $V_1, V_2, E_1, E_2 \subseteq \mathcal{P}^n(P)$, their intersections V, E also lie in $\mathcal{P}^n(P)$. Each element of V or E retains its original topological realization, and incidence relations are inherited. Thus (V, E) satisfies all axioms of a topological n-SuperHyperGraph. □

Theorem 3.17 (Restriction Theorem)**.** *Let $\mathrm{SuHyG}^{(n)} = (V, E)$ be a topological n-SuperHyperGraph on base P, and let $U \subseteq P$. Define*

$$V_U = \{\, v \in V : v \subseteq U\}, \quad E_U = \{\, e \in E : e \subseteq U\}.$$

Then, $\mathrm{SuHyG}_U^{(n)} = (V_U, E_U)$ is a topological n-SuperHyperGraph on base U.

Proof. Any $v \in V_U$ or $e \in E_U$ lies in $\mathcal{P}^n(U) \subseteq \mathcal{P}^n(P)$. We may restrict each Jordan region or point-collection realizing v or e to a small neighborhood around the points in U, yielding a valid topological realization on $\mathbb{R}^2$. Incidence and continuity are preserved under restriction, so (V_U, E_U) is a topological n-SuperHyperGraph on U. □

Theorem 3.18 (Minimal and Maximal Elements)**.** *In any finite topological n-SuperHyperGraph* $\mathrm{SuHyG}^{(n)} = (V, E)$*, the posets* $(V, \subseteq)$ *and* $(E, \subseteq)$ *each contain at least one minimal element and at least one maximal element.*

Proof. Because $\mathcal{P}^n(P)$ is finite, any nonempty subset of it has minimal and maximal elements under the inclusion order (by the well-ordering principle on finite sets). Thus V and E, being finite subsets, each admit at least one minimal and one maximal element. □

Theorem 3.19 (Nerve-Union Connectivity Theorem)**.** *Let* $\mathrm{SuHyG}^{(n)} = (V, E)$ *be a topological n-SuperHyperGraph whose superedges* $e \in E$ *are realized by connected regions* $R_e \subset \mathbb{R}^2$*. Form the nerve graph* $N(E)$ *with vertex set* E *and an edge between* e_i, e_j *if* $R_{e_i} \cap R_{e_j} \neq \emptyset$*. Then the union* $\bigcup_{e \in E} R_e$ *is path-connected if and only if* $N(E)$ *is connected.*

Proof. $\Rightarrow$: If $\bigcup_e R_e$ is path-connected, any two regions R_{e_i}, R_{e_j} can be joined by a continuous path lying entirely in the union. This path intersects a sequence of regions $R_{e_i} = R_{f_0}, R_{f_1}, \ldots, R_{f_k} = R_{e_j}$, so each consecutive pair meets nontrivially, yielding a path in $N(E)$.

$\Leftarrow$: If $N(E)$ is connected, for any two regions R_{e_i}, R_{e_j} there is a sequence $R_{f_0}, R_{f_1}, \ldots, R_{f_k}$ with $R_{f_\ell} \cap R_{f_{\ell+1}} \neq \emptyset$. Since each R_{f_ℓ} is connected, we can choose points $x_\ell \in R_{f_\ell} \cap R_{f_{\ell+1}}$ and join them by paths within each R_{f_ℓ}. Concatenating these paths gives a continuous path in $\bigcup_e R_e$ from a point in R_{e_i} to one in R_{e_j}. Hence the union is path-connected. □

Theorem 3.20 (Incidence Realization Theorem)**.** *In a topological n-SuperHyperGraph* $\mathrm{SuHyG}^{(n)} = (V, E)$*, for any* $v \in V$ *and* $e \in E$*,*

$$v \subseteq e \quad \Longleftrightarrow \quad \mathrm{Real}(v) \subseteq \mathrm{Real}(e),$$

where $\mathrm{Real}(\cdot)$ *denotes the assigned Jordan region or point-collection.*

Proof. ($\Rightarrow$) By definition of a topological realization, set inclusion $v \subseteq e$ implies that every point of v lies inside the region realizing e, hence $\mathrm{Real}(v) \subseteq \mathrm{Real}(e)$.

($\Leftarrow$) Conversely, if $\mathrm{Real}(v) \subseteq \mathrm{Real}(e)$ then each point of v (being part of $\mathrm{Real}(v)$) lies in the region for e, and by the Jordan curve construction no extra vertices can appear. Thus $v \subseteq e$ in the combinatorial sense. □

4. Conclusion and Future Works

In this paper, we introduced the concept of a *Topological n-SuperHyperGraph*, which unifies the structural hierarchy of n-SuperHyperGraphs with the geometric intuition of topological hypergraphs.

As future work, we aim to explore its potential applications in network analysis and to consider extensions using advanced uncertainty-handling frameworks such as Fuzzy Sets [46, 68], Intuitionistic Fuzzy Sets [5, 6], Neutrosophic Sets [3, 52, 65], Hesitant Fuzzy Sets [62, 67], and Plithogenic Sets [53, 60]. In addition, we expect further developments in the creation of programming tools based on Topological n-SuperHyperGraphs, deeper investigations into their mathematical structures, and progress in quantitative analyses and applied case studies.

Acknowledgements

The authors gratefully acknowledge the support, insights, and encouragement provided by colleagues, mentors, and reviewers. We appreciate the interest of our readers and the foundational work of scholars whose publications informed this study. We also thank the institutions and individuals who provided essential resources and infrastructure. Finally, we are indebted to everyone who offered assistance in any form.

Conflict of Interest

The authors declare that there are no competing interests related to this manuscript.

Funding

This research received no financial support from any external agencies or organizations.

Data Availability

This work is purely theoretical and does not rely on experimental or observational data. We invite future empirical investigations to validate and extend the concepts introduced here.

Ethical Approval

No ethical review was required, as this research does not involve human participants, animal subjects, or sensitive data.

Author Contributions

Takaaki Fujita and Florentin Smarandache conceived the study. Takaaki Fujita carried out the investigation and developed the methodology. The original draft was written by Takaaki Fujita, and both authors contributed to reviewing and editing.

Research Integrity

The authors confirm that this manuscript is their original work, has not been published elsewhere, and is not under consideration by another journal.

Use of Computational Tools

All proofs and derivations in this paper were performed manually. No computer algebra systems, automated theorem provers, or similar software (e.g., Mathematica, SageMath, Coq) were employed.

Scope and Accuracy

The theories and models presented herein have not yet been tested empirically. While every effort has been made to ensure correctness and proper citation, inadvertent errors may remain. Readers are encouraged to consult primary sources and provide feedback. The findings apply strictly within the stated assumptions and framework; their extension to other contexts requires further research. The views expressed are those of the authors alone and do not necessarily represent those of their affiliations.

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